DERIVATIONS OF HOMOTOPY ALGEBRAS

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ABSTRACT. We recall the definition of strong homotopy derivations of A_∞ algebras and introduce the corresponding definition for L_∞ algebras. We define strong homotopy inner derivations for both algebras and exhibit explicit examples of both.

1. INTRODUCTION

The concept of a strong homotopy derivation of an A_∞ algebra was introduced by Kajiura and Stasheff in [2]. In this note we will introduce the corresponding concept for L_∞ algebras. We will discuss several concrete examples of such algebras and strong homotopy inner derivations on them.

In section 2, we recall the definitions of A_∞ and L_∞ algebras and discuss an explicit example of each. We will use these examples to exhibit examples of strong homotopy derivations.

In section 3, we review the definition of a strong homotopy derivation of an A_∞ algebra. We introduce the concept of an inner such derivation of these algebras and present an explicit example of this concept by using the A_∞ algebra in the previous section.

The next section contains our definition of a strong homotopy derivation of an L_∞ algebra. We discuss the concept of inner derivation and present a concrete example of such a derivation on the L_∞ algebra in Section 2.

In the final section we will discuss the relationship between the A_∞ data and the L_∞ data using symmetrization.

We work in the setting of $\mathbb{Z}$ graded vector spaces and will occasionally use the notation $|x|$ to denote the degree of an element x.

2. A_∞ AND L_∞ ALGEBRAS

Definition 1. *An A_∞ algebra* [6] *structure on a $\mathbb{Z}$ graded vector space V is a collection of degree one linear maps $m_n : V^{\otimes n} \to V$ that satisfy the relations*

$$\sum_{k+l=n+1} \sum_{i=1}^{k} (-1)^\alpha m_k(v_1, \ldots, v_{i-1}, m_l(v_i, \ldots, v_{i+l-1}), v_{i+l}, \ldots, v_n) = 0 \qquad (2.1)$$

for $n \geq 1$ and α is the sum of the degrees of the elements $v_1, \ldots, v_{i-1}$.

We remark that this definition differs from but is equivalent to the original definition [6] in which the maps m_n have degree $n - 2$ and the signs are adjusted accordingly. It is well known [6] that the structure maps m_n's may be extended to a degree $+1$ coderivation $\boldsymbol{m}$ on the tensor coalgebra $T^c(V)$ of V, and that the relations are equivalent to the equation $\boldsymbol{m}^2 = 0$.

Example 2. [1] *Consider the graded vector space in which* V_{-1} *has basis* $< x_1, x_2 >$, V_0 *has basis* $< y >$, *and* $V_n = 0$ *otherwise. Define degree one maps*

$$m_1(x_1) = m_1(x_2) = y$$
$$m_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) = x_1, \quad 0 \le k \le n - 2$$
$$m_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = x_1$$
$$m_n(x_1 \otimes y^{\otimes n-1}) = y$$

and $m_n = 0$ *on the remaining elements of* V. *This determines an* A_∞ *algebra structure on* V.

We next recall the definition of L_∞ algebras.

Definition 3. [5] *An* L_∞ *algebra structure on a* $\mathbb{Z}$ *graded vector space* V *is a collection of degree one graded symmetric linear maps* $l_n : V^{\otimes n} \to V, n \ge 1$, *that satisfy the relations (higher order Jacobi relations)*

$$\sum_{j+k=n} \sum_{\sigma} (-1)^{e(\sigma)} l_{1+j}(l_k(v_{\sigma(1)}, \dots, v_{\sigma(k)}), v_{\sigma(k+1)}, \dots, v_{\sigma(n)}) = 0$$

where σ *runs over all (k, n-k) unshuffle permutations. The exponent* $e(\sigma)$ *is the sum of the products of the degrees of the elements that are permuted, sometimes known as the Koszul sign..*

Again, we remark that this definition differs from but is equivalent to the original definition [5] in which the maps l_n have degree $n - 2$ and are graded skew symmetric with the signs adjusted. Also, the structure maps may be extended to a degree $+1$ coderivation $\boldsymbol{l}$ on the symmetric coalgebra $S^c(V)$ on V, and the relations are equivalent to $\boldsymbol{l}^2 = 0$. [4],[5]

It is well known that skew symmetrization of an A_∞ algebra structure yields an L_∞ algebra structure [4] when one utilizes the original definitions. However, with the definitions that we use here, we symmetrize the data in the example above to obtain

Example 4. *Consider the graded vector space in which* V_{-1} *has basis* $< x_1, x_2 >$, V_0 *has basis* $< y >$, *and* $V_n = 0$ *otherwise. Define degree one symmetric maps*

$$l_1(x_1) = l_1(x_2) = y$$
$$l_n(x_1 \otimes y^{\otimes n-1}) = (n - 1)! \, y$$
$$l_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n - 2)! \, x_1$$

and extend the maps using symmetry. This yields an L_∞ *algebra structure on* V.

We will use these examples above to illustrate examples of strong homotopy derivations which we will define in the next two sections.

3. STRONG HOMOTOPY DERIVATIONS OF A_∞ ALGEBRAS

Kajiura and Stasheff [2] have formulated the following definition:

Definition 5. *A strong homotopy derivation of degree one of an A_∞ algebra $(V, \{m_n\})$ is a collection of degree one linear maps $\theta_q : V^{\otimes q} \to V, q \geq 1$, that satisfy the relations*

$$0 = \sum_{r+s=q+1} \sum_{i=0}^{r-1} (-1)^{\beta(s,i)} \theta_r(v_1, \ldots, v_i, m_s(v_{i+1}, \ldots, v_{i+s}), \ldots, v_q) \qquad (3.1)$$

$$+(-1)^{\beta(s,i)} m_r(v_1, \ldots, v_i, \theta_s(v_{i+1}, \ldots, v_{i+s}), \ldots, v_q).$$

The exponent $\beta(s,i)$ results from moving the degree one maps m_s and θ_s past $(v_1, \ldots, v_i)$. The θ_q's may be extended to a degree $+1$ coderivation $\boldsymbol{\theta}$ on $T^c(V)$ and the relations then can be described by the equation $[\boldsymbol{m}, \boldsymbol{\theta}] = 0$.

As an example of such a structure, we can define a strong homotopy inner derivation of an A_∞ algebra.

Proposition 6. *Let $(V, \{m_n\})$ be an A_∞ algebra and let $a \in V$ have the property that $m_1(a) = 0$ and the degree of a is even. Then the maps*

$$\theta_n(v_1, \ldots, v_n) = m_{n+1}(a, v_1, \ldots, v_n) + \ldots \qquad (3.2)$$

$$+m_{n+1}(v_1, \ldots, v_i, a, v_{i+1}, \ldots, v_n) + \cdots + m_{n+1}(v_1, \ldots, v_n, a)$$

define a strong homotopy derivation of V. We call such a derivation inner.

Proof. It can be calculated that the defining relations for a strong homotopy derivation in this case result in $n + 1$ copies of the defining relations for an A_∞ algebra except for terms that involve $m_1(a)$. Because of our requirement that $m_1(a) = 0$, we may add in the missing terms and utilize the A_∞ algebra relations $n + 1$ times to obtain the result. $\qquad \square$

Recall the example of an A_∞ algebra in Section 2. There we had the following data. Consider the graded vector space in which V_{-1} has basis $< x_1, x_2 >$, V_0 has basis $< y >$, and $V_n = 0$ otherwise. We may construct a strong homotopy inner derivation on V by letting $a = y$. One may then calculate resulting θ_n's to be

$$\theta_1(x_1) = y$$

$$\theta_n(x_1 \otimes y^{\otimes k} \otimes x_1 \otimes y^{\otimes n-2-k}) = nx_1$$

$$\theta_n(x_1 \otimes y^{\otimes n-1}) = ny$$

$$\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-1)x_1$$

and $\theta_n = 0$ on the terms not mentioned.

4. Strong Homotopy Derivations of L_∞ Algebras

We now turn our attention to L_∞ algebras.

Definition 7. *A strong homotopy derivation of degree one of an L_∞ algebra $(V, \{l_n\})$ is a collection of degree one graded symmetric linear maps $\theta_q : V^{\otimes q} \to V, q \geq 1$, that satisfy the relations*

$$\sum_{j=1}^{n} \sum_{\sigma} (-1)^{e(\sigma)} \theta_{n-j+1}(l_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)}) \qquad (4.1)$$

$$+ (-1)^{e(\sigma)} l_{n-j+1}(\theta_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)}) = 0$$

where σ runs over all $(j, n-j)$ unshuffle permutations.

The exponent $e(\sigma)$ is the sum of the products of the degrees of the permuted elements. As we saw for A_∞ derivations, we may express the defining relations for strong homotopy derivations on L_∞ algebras by the equation $[\boldsymbol{l}, \boldsymbol{\theta}] = 0$ where $\boldsymbol{\theta}$ is the degree $+1$ coderivation on $S^c(V)$ induced by the θ_n's. See [7] for details.

As an example, we define a strong homotopy inner derivation of an L_∞ algebra.

Proposition 8. *Let $(V, \{l_n\})$ be an L_∞ algebra and let $a \in V$ have the property that $l_1(a) = 0$ and the degree of a is even. Then the maps*

$$\theta_n(v_1, \ldots, v_n) = l_{n+1}(v_1, \ldots, v_n, a) \qquad (4.2)$$

define a strong homotopy derivation of V.

Proof. We compute

$$\sum_{\substack{j=1 \\ \sigma}}^{n} (-1)^{e(\sigma)} \theta_{n-j+1}(l_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)})$$

$$+ (-1)^{e(\sigma)} l_{n-j+1}(\theta_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)})$$

$$= \sum_{\substack{j=1 \\ \sigma}}^{n} (-1)^{e(\sigma)} l_{n-j+2}(l_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)}, a)$$

$$+ (-1)^{e(\sigma)} l_{n-j+1}(l_{j+1}(v_{\sigma(1)}, \ldots, v_{\sigma(j)}, a), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)})$$

$$= \sum_{\substack{j=1 \\ \sigma}}^{n} (-1)^{e(\sigma)} l_{n-j+2}(l_j(v_{\sigma(1)}, \ldots, v_{\sigma(j)}), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)}, a)$$

$$+ (-1)^{e(\sigma)}(-1)^{\alpha} l_{n-j+1}(l_{j+1}(v_{\sigma(1)} \ldots, v_{\sigma(j)}, a), v_{\sigma(j+1)}, \ldots, v_{\sigma(n)})$$

$$+ (-1)^{\beta} l_{n+1}(l_1(a), v_1, \ldots, v_n) = 0$$

because these are precisely the L_∞ algebra relations on $(v_1, \ldots, v_n, a)$. Note that it is necessary to add the last line, where $\beta = |a| \sum_{i=1}^{n} |v_i|$, to the homotopy derivation relations to obtain the L_∞ algebra relations; this term, however, is zero because of our assumptions on the element a. The sign in the next to last line reduces to the required $(-1)^{e(\sigma)}$ because $\alpha = |a| \sum_{i=j+1}^{n} |v_{\sigma(i)}|$ is even. $\square$

Recall the example of an L_∞ algebra in Section 2. There, $V = V_{-1} \oplus V_0$ with basis for $V_{-1} = <x_1, x_2>$ and basis for $V_0 = <y>$ and the degree one graded symmetric maps given by

$$l_1(x_1) = l_1(x_2) = y$$
$$l_n(x_1 \otimes y^{\otimes n-1}) = (n-1)!\, y$$
$$l_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-2)!\, x_1.$$

We construct a strong homotopy derivation of V by letting $a = y$ and then calculate the resulting θ_n's to be

$$\theta_1(x_1) = y$$
$$\theta_n(x_1 \otimes y^{\otimes n-1}) = n!\, y$$
$$\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) = (n-1)!\, x_1$$

and θ_n is zero on the elements not listed.

For example,

$$\theta_n(x_1 \otimes y^{\otimes n-2} \otimes x_2) := l_{n+1}(x_1 \otimes y^{\otimes n-2} \otimes x_2 \otimes y)$$
$$= l_{n+1}(x_1 \otimes y^{\otimes n-1} \otimes x_2) = (n-1)!\, x_1.$$

5. Symmetrization of A_∞ Derivations

We recall that there is a well known injective coalgebra map $\chi : S^c(V) \longrightarrow T^c(V)$ given by

$$\chi(v_1, \ldots, v_n) = \sum_{\sigma \in S_n} (-1)^{e(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

where $(-1)^{e(\sigma)}$ is the Koszul sign.

Suppose that $f : T^c(V) \longrightarrow V$ is a linear map which extends to the coderivation $\boldsymbol{f} : T^c(V) \longrightarrow T^c(V)$ such that $\pi_1 \circ \boldsymbol{f} = f$, where $\pi_1 : T^c(V) \longrightarrow V$ is projection. Then the linear map $f \circ \chi : S^c(V) \longrightarrow V$ extends to the coderivation $\boldsymbol{f} \circ \boldsymbol{\chi} : S^c(V) \longrightarrow S^c(V)$ and the following diagram commutes (Prop. 5, [3])

$$
\begin{array}{ccccc}
S^c(V) & \xrightarrow{\chi} & T^c(V) & & \\
\boldsymbol{f} \circ \boldsymbol{\chi} \uparrow & & \boldsymbol{f} \uparrow & \searrow \pi_1 & \\
S^c(V) & \xrightarrow{\chi} & T^c(V) & \xrightarrow{f} & V
\end{array}
$$

The symmetrization of an A_∞ algebra structure that was mentioned in Section 2 may then be described by the commutative diagram

$$
\begin{array}{ccccc}
S^c(V) & \xrightarrow{\chi} & T^c(V) & & \\
\boldsymbol{l} \uparrow & & \boldsymbol{m} \uparrow & \searrow \pi_1 & \\
S^c(V) & \xrightarrow{\chi} & T^c(V) & \xrightarrow{m} & V
\end{array}
$$

where $m = \sum m_n : T^c(V) \longrightarrow V$ is the collection of the A_∞ algebra structure maps, $\boldsymbol{m}$ is the lift of m to a coderivation on $T^c(V)$ with $\boldsymbol{m}^2 = 0$, and the L_∞ algebra structure $\boldsymbol{l}$ is the lift of the map $m \circ \chi : S^c(V) \longrightarrow V$ to a coderivation on $S^c(V)$.

We now address the issue of symmetrization of strong homotopy derivations of A_∞ algebras.

Proposition 9. *Let $\theta = \{\theta_n\}$ denote the the collection of maps giving a strong homotopy derivation on the A_∞ algebra (V, m). Regard θ as a map $T^c(V) \longrightarrow V$ and lift it to the coderivation $\boldsymbol{\theta}$ on $T^c(V)$. Then the extension of the map $\theta \circ \chi : S^c(V) \longrightarrow V$ to the coderivation $\boldsymbol{\theta}'$ on $S^c(V)$ is a strong homotopy derivation on the L_∞ algebra V with algebra structure given by $m \circ \chi$.*

Proof. We claim that $[\boldsymbol{l}, \boldsymbol{\theta}'] = 0$. We have the commutative diagram

$$
\begin{array}{ccccc}
S^c(V) & \xrightarrow{\chi} & T^c(V) & & \\
\boldsymbol{\theta}' \uparrow & & \boldsymbol{\theta} \uparrow & \searrow \pi_1 & \\
S^c(V) & \xrightarrow{\chi} & T^c(V) & \xrightarrow{\theta} & V
\end{array}
$$

and we calculate

$$
\begin{aligned}
\chi[\boldsymbol{l}, \boldsymbol{\theta}'] &= \chi(\boldsymbol{l}\boldsymbol{\theta}' + \boldsymbol{\theta}'\boldsymbol{l}) \\
&= (\chi\boldsymbol{l})\boldsymbol{\theta}' + (\chi\boldsymbol{\theta}')\boldsymbol{l} \\
&= \boldsymbol{m}(\chi\boldsymbol{\theta}') + \boldsymbol{\theta}(\chi\boldsymbol{l}) \\
&= \boldsymbol{m}\boldsymbol{\theta}\chi + \boldsymbol{\theta}\boldsymbol{m}\chi \\
&= [\boldsymbol{m}, \boldsymbol{\theta}]\chi = 0
\end{aligned}
$$

because $\chi \circ \boldsymbol{l} = \boldsymbol{m} \circ \chi$ from the commutative diagram and $[\boldsymbol{m}, \boldsymbol{\theta}] = 0$ because $\boldsymbol{\theta}$ is a strong homotopy derivation of an A_∞ algebra. Because χ is injective, it follows that $[\boldsymbol{l}, \boldsymbol{\theta}'] = 0$. □

We thank the referee for the careful reading of this article and for the helpful corrections.

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